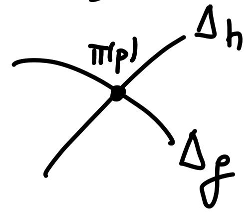
§14. Invariants of SNC abelian coverings of surfaces

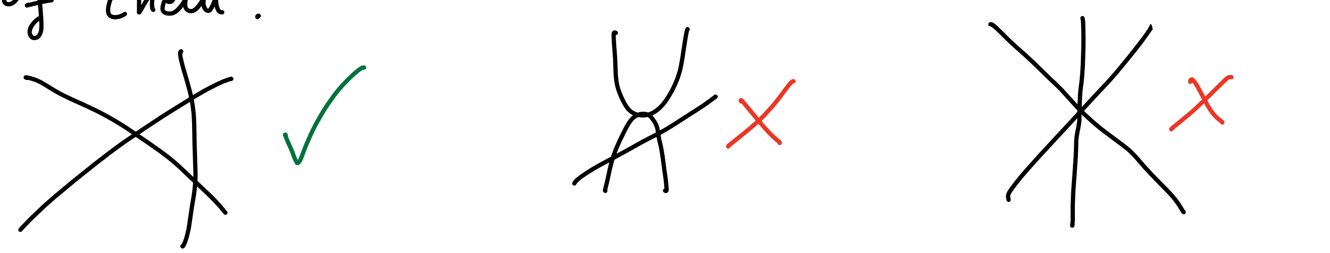
We want to study some singular abelian covers of surfaces.
We have seen that given $\pi: X \to Y$ ab. cover of a $\boxed{\text{surface}}$, then $p \in X$ is smooth if and only if

- $\pi(p)$ is smooth;
- $\pi(p)$ belongs to at most two irreducible components of some D_g and D_h, and D_g, D_h intersects transversally at $\pi(p)$;
- $\langle g \rangle \oplus \langle h \rangle \to G$ is an injective map.



We want to study what kind of singularities arises if we delete the third condition, namely $\langle g \rangle \oplus \langle h \rangle \to G$ is injective.

Def A Simple normal crossing (SNC) abelian cover $\pi: X \to Y$ is an abelian cover with a reduced branch locus D consisting of smooth irreducible components intersecting two by two transversally and for which no point of Y belongs to more than two of them.



Remark From the smoothness properties, then the singular locus of X lies on

$$\Delta := \{ q \in Y \mid q \in D_g \cap D_h, \; g, h \in G, \text{ and } \langle g \rangle \oplus \langle h \rangle \to G \text{ not injective} \}$$

(namely, $\pi(\mathrm{Sing}(X)) = \Delta$).

First of all, let us study these kind of singularities.

Prop (Pardini, Prop 3.3.)

Let $p \in X$ be a point over $q \in \Delta$, so $q \in D_g \cap D_h$ and $\langle g \rangle \oplus \langle h \rangle \to G$ is not injective, for some $g, h \in G$.

Let us consider $n := |\langle g \rangle \cap \langle h \rangle|$ and let $1 \leq s \leq |h| - 1$ be the lowest integer such that $h^s = g^a$, $1 \leq a \leq |g| - 1$.

Then p is a cyclic quotient singularity of type

$$\boxed{\frac{1}{n}\left(1, n - \frac{na}{|g|}\right)}$$

proof We distinguish three cases:

Case 1: $\langle g \rangle = \langle h \rangle$ and $h = g$. We take the intermediate quotient $X \xrightarrow{\pi'} X/\langle g \rangle \xrightarrow{G/\langle g \rangle} Y$, so $X/\langle g \rangle \to Y$ is étale locally around q and we can use π' to study the kind of singularity of p. Let χ be the dual character of g ($\chi(g) = e^{\frac{2\pi i}{n}}$), so χ generates $\langle g \rangle^*$.

We observe that

$$w_x^k = w_{x^k}\cdot\Big(\prod_{t\in\langle g\rangle} \sigma_t^{q^t_{x^k}}\Big)$$

where $q^t_{x^k} = \Big\lfloor \frac{k\, r^t_x}{|t|} \Big\rfloor$. For $t=g$, then $r^g_x = 1$ so $q^g_{x^k}=0$, while for $t\neq g$, then $\sigma_t^{q^t_{x^k}}$ would not vanish at q, so locally around p $\prod_{t\in\langle g\rangle} \sigma_t^{q^t_x} \neq 0$ and then w_{x^k} is a redundant variable.

So we only have one variable locally around p, w_x, and only one equation:

$$w_x^{|g|} = \sigma_g = x\cdot y$$

where $\{x=0\}$ and $\{y=0\}$ are the local parameter of the pair of irred. comp. over D_g intersecting transversally at p. Thus, locally around p, X is the zero locus

$$z^m = x\cdot y$$

whose origin is exactly a cyclic quotient singularity of type $\frac{1}{m}(1, m-1)$, as we want.

(to prove this, you must show that $\mathbb{C}^2/(\mathbb{Z}/m)$ is isomorphic to $Z(z^m - x\cdot y) \subseteq \mathbb{C}^3$. Look at the ring of invariant polynomials).

<u>Case 2</u>: $\langle g\rangle = \langle h\rangle$ but $h = g^d$, $2 \leq d \leq |g|-1$.

Locally around p we have

$$w_x^{|g|} = \prod_{t\in\langle g\rangle} \sigma_t^{r^t_x} = \sigma_h^d \cdot \sigma_g \cdot (\text{something not vanishing at } p)$$

we can delete this locally around p.

Let $k := \min\{\alpha\in\{1,\dots,|g|-1\} \mid \alpha a > |g|\}$, then

$$w_x^k = w_{x^k}\cdot \prod_{t\in\langle g\rangle} \sigma_t^{q^t_{x^k}} = w_{x^k}\left(\sigma_h^{q^h_{x^k}}\cdot \sigma_g^{q^g_{x^k}}\right) = w_{x^k}\cdot \sigma_h^{\lfloor \frac{ka}{|g|}\rfloor}$$

remind $q^t_{x^k} = \left\lfloor \frac{k v^t_x}{|t|}\right\rfloor$

Instead, for $1\le\alpha<k$, we have

$$w_x^\alpha = w_{x^\alpha}\cdot \underbrace{\sigma_h^{\lfloor \frac{\alpha a}{|g|}\rfloor}}_{=1} = w_{x^\alpha}$$ ← so they are redundant variables.

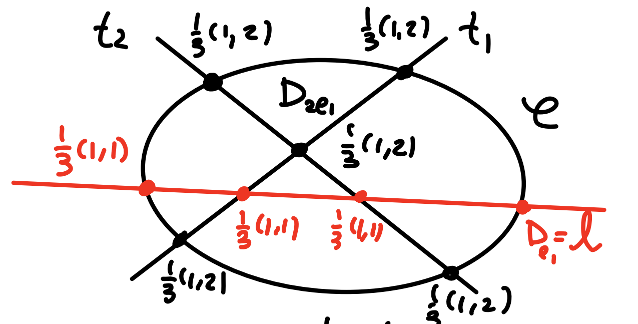
Using a similar approach as above, we obtain a set of equations whose it is know in the literature tha the origin gives a singularity of type $\frac{1}{n}(1,n-a)$.

Case 3 $\langle g\rangle \neq \langle h\rangle$, $h^s=g^a$. Similar to the case 2. ▨

Example Consider $G=\mathbb{Z}_3^{\langle e_1\rangle}$ and $\pi: X \dashrightarrow \mathbb{P}^2$ defined by

$D_{e_1} := \ell$, $D_{2e_1} = t_1+t_2+\mathcal{C}$, ℓ, t_i lines, $\mathcal{C}$ smooth conic.

$3\mathcal{L}_{\varepsilon_1} = \ell + 2(t_1+t_2+\mathcal{C}) \equiv 9H \Rightarrow \mathcal{L}_{\varepsilon_1} \equiv 3H$ ✓ so $\pi: X\dashrightarrow\mathbb{P}^2$ exists.



Let $q\in D_{e_1}\cap D_{2e_1}$. Then

$2e_1 = 2\cdot e_1$, so $n = |\langle e_1\rangle \cap \langle 2e_1\rangle| = 3$,

$a=2$, and any point point p over q (which are in total $\frac{|G|}{|Stab(p)|} = \frac{3}{3} = 1$) is a cyclic quot. sing. of type $\frac{1}{3}(1,3-2)$

Instead, consider $q \in t_1 \cap t_2$. Then $n = |\langle e_1 \rangle \cap \langle e_2 \rangle| = 3$, $a = 1$, and so the point over q is of type $\frac{1}{3}(1,2)$.

Resolution of cyclic quotient singularities

For the rest of the note $\boxed{X \text{ has dimension } 2}$ and it is normal.

Given a singular point $p \in X$, a resolut. of p is a map $\tilde{X} \xrightarrow{b} X$ such that

$$\tilde{X} \setminus b^{-1}(p) \longrightarrow X \setminus b \quad \text{is an isomorphism}$$

and $\tilde{X}$ is smooth along $b^{-1}(p)$.

A resolution of X is the resolution of all its singularities, which is then a smooth surface.

Thm (Italian School XX, Walker-Zariski, Hironaka)

Idea in dim 2 / Proofs in dim 2 / dim ≥ 3

Any singular surface admits a resolution.

Def We say that $\tilde{X} \xrightarrow{b} X$ is a minimal resolution of singularities if given a resolution $\tilde{X}' \xrightarrow{b'} X$, then b' factorizes

$$\begin{tikzcd}
\tilde{X}' \arrow[rr, "b'"] \arrow[dr, "\text{birational}"'] & & X \\
& \tilde{X} \arrow[ur, "b"'] &
\end{tikzcd}$$

Warning: Do not confuse "the minimal resolution" with "a minimal surface". Often a minimal resolution of singularities is NOT a minimal surface, in the sense that it may have (-1)-curves.

Thm X admits a minimal resolution of singularities, and it is unique!!

We can finally show how to resolve a cyclic quotient singularity.

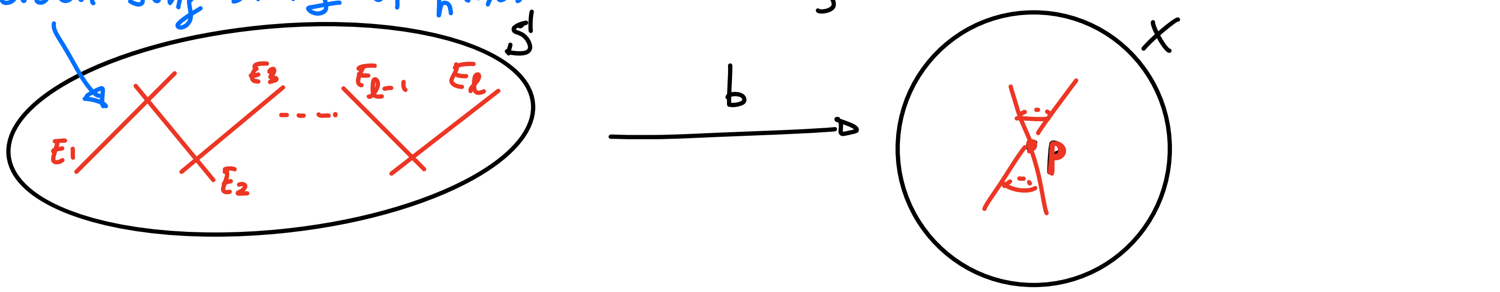
Thm Let $p \in X$ be a cyclic quotient singularity of type $\frac{1}{n}(1,a)$. Let us consider the continued fraction

$$\frac{n}{a} := [b_1, \dots, b_\ell] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\dots}}}$$

It there exists a resolution of p, $S \xrightarrow{b} X$ s.t. $b^{-1}(p) = E_1 \cup \dots \cup E_\ell$, where E_i is a smooth rational curve ($E_i \simeq \mathbb{P}^1$) and

$$E_i^2 = -b_i \quad , \quad \begin{array}{l} E_i E_{i+1} = 1 \;\; \forall i = 1, \dots \ell - 1 \\ E_i E_j = 0 \;\; \text{otherwise} \end{array}$$

It is called Hirzebruch-Jung string of $\frac{1}{n}(1,a)$



Thm Given X with only isolated cyclic quot. singularities, let $S \xrightarrow{b} X$ be the resolution obtained in the previous thm recursively on each isolated singularity. Then S is the minimal res. of singularities of X.

Invariants of the minimal res. of singularities

X normal, we denote by K_X the canonical Weil divisor on X, namely (the closure) of $i_*(\Omega^n_{X^\circ})$, $i: X^\circ \hookrightarrow X$ being the inclusion of the smooth locus of X.

Assume X is $\mathbb{Q}$-Cartier (namely mK_X is Cartier for some $m \in \mathbb{N}$).
Let m be the index of X, i.e. the smallest positive integer s.t. mK_X is Cartier.
Let $S \xrightarrow{b} X$ be the minimal resolution of sing. of X. Then

$$\boxed{mK_S = b^*(mK_X) + \sum a_i E_i} \qquad (*)$$

where E_i are the exceptional divisors of the resolution, and a_i are integer numbers, that we still need to determine.

Let us assume that all the singular. are cyclic quotients, so that $\sum a_i E_i = \sum_{p \in \mathrm{Sing}(X)} \left(\sum a_i E_i^{(p)} \right)$

where $E_1^{(p)} \cup \cdots \cup E_\ell^{(p)}$ is the H-J string of p.

Thm The coefficients $a_1, \dots a_\ell$ of a HJ string of type $\frac{1}{n}(1,a)$, $\frac{n}{a} = [b_1, \dots b_\ell]$, appearing in (*) verify the linear system

$$\begin{cases} -a_1 b_1 + a_2 = m(b_1 - 2) \\ a_1 - a_2 b_2 + a_3 = m(b_2 - 2) \\ \vdots \\ a_{\ell-1} - b_\ell a_\ell = m(b_\ell - 2) \end{cases}$$

proof Locally around p we can write

$$m k_S \equiv b^*(m k_X) + \sum_{i=1}^{\ell} a_i E_i^{(p)}$$

where $E_1^{(p)} \cup \cdots \cup E_\ell^{(p)}$ is the HJ- string of p.

$$\underbrace{m k_S \cdot E_J^{(p)}}_{\substack{m[2g(E_J^{(p)}) - 2 - (E_J^{(p)})^2] \\ \| \\ m[b_J - 2]}} = \overbrace{b^*(m k_X) \cdot E_J^{(p)}}^{0} + \underbrace{\left(\sum_{i=1}^{\ell} a_i E_i^{(p)} \right) E_J^{(p)}}_{a_{J-1} - a_J b_J + a_{J+1}}$$

(remember that on a smooth surf. S, $2g(C) - 2 = C^2 + C k_S$ for any irreduc. curve of S)

$$\Rightarrow \begin{cases} -a_1 b_1 + a_2 = m(b_1 - 2) \\ a_1 - a_2 b_2 + a_3 = m(b_2 - 2) \\ \vdots \\ a_{\ell-1} - b_\ell a_\ell = m(b_\ell - 2) \end{cases}$$

▨

Prop (F. 2026)
There is a closed formula for the solution of the above linear system. If p is of typ $\frac{1}{n}(1,a)$, $\frac{n}{a}=[b_1,\dots b_l]$, then we denote by $\frac{n_i}{c_i}:=[b_i, b_{i+1},\dots,b_l]$, and $\frac{m_i}{d_i}=[b_1,b_2,\dots b_i]$ the i-th truncated continued fractions of $\frac{n}{a}$. Then

$$\boxed{a_i = c_i + [d_i]^{-1}_{\mathrm{mod}\, m_i} - m}$$

Thus, we finally have

$$m k_S \equiv b^*(m k_X) - \left[\sum_{x\in \mathrm{Sing}(X)} \left(\sum_{i=1}^{l_x} m - (c_i + [d_i]^{-1}_{\mathrm{mod}\, m_i})\right)\right]$$

Thm Let X be a norm. surf with at most isol. cyclic quot. sing and let $S \xrightarrow{b} X$ be the min. resolution of sing. of X. Then

$$\boxed{e(S) = e_c(X^\circ) + \sum_{p\in \mathrm{Sing}(X)} (l_p + 1)}$$

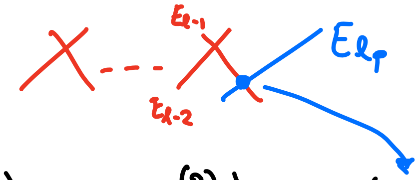
where l_p is the length of the Hirzebruch-Jung string of the resolution of p.

proof We apply inclusion-exclusion principle to the Euler characteristic on compact support:

$$e(S) = e_c(S) = e_c(S \setminus b^{-1}(\mathrm{Sing}(S)) + e_c(b^{-1}(\mathrm{Sing}(S)))$$

$$= e_c(X^\circ) + \sum_{p \in \mathrm{Sing}(X)} e_c(\underbrace{b^{-1}(p)}_{= E_1^{(p)} \cup \dots \cup E_{\ell_p}^{(p)}})$$

so we need to compute the Euler Characteristic of a Hirzebruch-Jung string



$$e_c(E_1^{(p)} \cup \dots \cup E_{\ell_p}^{(p)}) = e_c(E_1^{(p)} \cup \dots \cup E_{\ell_p - 1}^{(p)}) + \underbrace{\underbrace{e_c(E_{\ell_p}^{(p)})}_{=2} - \underbrace{e_c(pt)}_{=1}}_{1}$$

$$= \dots = e_c(E_1^{(p)} \cup E_2^{(p)}) + (\ell_p - 2)$$

$$= \underbrace{e_c(E_1^{(p)})}_{=2} + \underbrace{e_c(E_2^{(p)})}_{=2} - 1 + (\ell_p - 2)$$

$$= \ell_p + 1 \qquad \blacksquare$$

We can finally prove

Thm Let $\pi: X \to Y$ be a SNC ab. covering with Galois group G building data $\{\mathcal{L}_\chi\}_{\chi \in G^*}, \{D_g\}_{g \in G}$. Let $b: S \to X$ be the minimal res. of singul. of X. We define $\tilde{\pi}: S \to Y$, $\tilde{\pi} := \pi \circ b$. Then:

$$p_g(S) = \sum_{\chi \in G^*} h^0(Y, K_Y + \mathcal{L}_\chi), \qquad q(S) = \sum_{\chi \in G^*} h^1(Y, -\mathcal{L}_\chi)$$

$$\chi(\mathcal{O}_X) = |G|\,\chi(\mathcal{O}_Y) + \frac{1}{2} \sum_{\chi \in G^* \setminus 1} \mathcal{L}_\chi(\mathcal{L}_\chi + K_Y)$$

$$e \cdot K_S = \tilde{\pi}^*\left(e \cdot K_Y + \sum_{g \in G} \frac{e \cdot (|g| - 1)}{|g|} D_g \right) - \sum_{q \in \Lambda} \left(\sum_{\substack{p \in \pi^{-1}(q) \\ \text{type } \frac{1}{n}(1,a)}} \left(\sum_{i=1}^{\ell_p} e \cdot \left(1 - \frac{(c_i + [d_i]^{-1})_{\mathrm{mod}\, w_i}}{n} \right) E_i^p \right) \right)$$

$e :=$ exponent of $|G|$

$$K_S^2 = \left(K_Y + \sum_{g\in G} \frac{|g|-1}{|g|} D_g\right)^2 + \sum_{q\in \Lambda}\left(\sum_{p\in\pi^{-1}(q)}\left(\frac{2+a+[a]^{-1}_n}{n} - 2 + \sum_{i=1}^{l_p}(b_i-2)\right)\right)$$

$$e(S) = |G|\left[e(Y) - \sum_{g\in G}\left(1-\frac{1}{|g|}\right)e(D_g) + \frac{1}{2}\sum_{\Delta_g\neq\Delta_h}\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)\Delta_g\cdot\Delta_h\right]$$

$$+ \frac{1}{2}\sum_{\Delta_g\neq\Delta_h}\underbrace{\left(\frac{|G|}{|\langle g,h\rangle|} - \frac{|G|}{|g|\cdot|h|}\right)}_{\text{notice that this is zero for } \langle g\rangle\cap\langle h\rangle = 1_G}\Delta_g\cdot\Delta_h + \sum_{q\in\Lambda}\left(\sum_{p\in\pi^{-1}(q)} l_p\right)$$

$$k(S) \leq k\left(Y, |G|K_Y + \sum_{g\in G}\frac{|G|(|g|-1)}{|g|} D_g\right)$$

proof By Freitag thm, every holomorphic k-form of the smooth locus of X extedends uniquely to a global holomorphic k-form of the minimal res. of singularities (if X has at most cyclic quotient singularities). This implies

$$h^0(\Omega^i_S) = h^0(\Omega^i_{X^\circ}) \underset{\substack{\downarrow\\ \text{it is } h^{i,0}(\theta_{X^\circ}) = h^{0,i}(\theta_{X^\circ}) \\ \text{by Hodge symmetry}}}{=} h^i(\theta_{X^\circ}) = \sum_{\chi\in G^*} h^i(-L_\chi)$$

$X^\circ \to Y\setminus\Delta$ is an ab. cover with group G branched on $\sum_i D_g\setminus\Delta$

This implies $q(S) = h^0(\Omega^1_S) = \sum_{\chi\in G^*} h^1(-L_\chi)$ and

$$p_g(S) = h^0(\Omega^2_S) = \sum_{\chi\in G^*} h^2(-L_\chi) \underset{\substack{\downarrow\\ \text{Serre Duality}}}{=} \sum_{\chi\in G^*} h^0(K_Y + L_\chi)$$

Using the same proof as in the case X is smooth (see the theorem on the invariants of the previous lectures) we can conclude directly

$$\chi(\vartheta_S) = |G|\chi(\vartheta_Y) + \frac{1}{2}\sum_{\chi \in G^* \setminus 1} L_\chi(L_\chi + K_Y)$$

Instead, the formula for $e \cdot k_S$ follows directly from the previous thm and the fact that $e \cdot K_X$ is Cartier and it is given by:

$$e \cdot k_{X^\circ} \underset{\downarrow}{=} \pi^*\Big(e \cdot k_Y + \sum_{g \in G} \frac{e(|g|-1)}{|g|} D_g\Big) \quad \text{on } X^\circ$$

it follows from the formula of $|G| k_X$ in the smooth case.

so by taking the closure on X we obtain

$$e \cdot k_X = \pi^*\Big(e \cdot k_Y + \sum_{g \in G} \frac{e \cdot (|g|-1)}{|g|} D_g\Big)$$

It only remains to determine $e(S)$:

$$e(S) = e_c(X^\circ) + \sum_{q \in \Lambda} \Big(\sum_{p \in \pi^{-1}(q)} (l_p + 1)\Big)$$

$$\overset{\shortparallel}{\sum_{q \in \Lambda}\Big(\sum_{p \in \pi^{-1}(q)} l_p\Big) + \sum_{q \in \Lambda} |\pi^{-1}(q)|} = \sum_{q \in \Delta}\Big(\sum_{p \in \pi^{-1}(q)} l_p\Big) + \frac{1}{2}\sum_{\substack{\Delta_g \neq \Delta_h \\ \langle g \rangle \cap \langle h \rangle \neq 1_G}} \frac{|G|}{|\langle g, h\rangle|} \Delta_g \cdot \Delta_h$$

We observe that the computation of $e_c(X^\circ)$ can be done using the fact $\pi: X^\circ \to Y \setminus \Lambda$ is an ab. covering branched over $D = \sum_{g \in G} D_g \setminus \Lambda$ as we already done in the smooth case. Thus

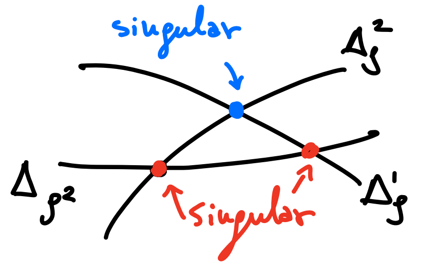
$$e_c(X^\circ) = |G|\Big[e_c(Y \setminus \Lambda) - \sum_{g \in G}\Big(1 - \frac{1}{|g|}\Big) e_c(D_g \setminus \Lambda) + \frac{1}{2}\sum_{g \neq h}\Big(1 - \frac{1}{|g|}\Big)\Big(1 - \frac{1}{|h|}\Big) D_g \setminus \Lambda \cdot D_h \setminus \Lambda\Big]$$

$$e(Y) = e_c(Y) = e_c(Y \setminus \Lambda) + e_c(\Lambda) = e_c(Y \setminus \Delta) + |\Lambda|$$

$$\Rightarrow \quad e_c(Y \setminus \Lambda) = e(Y) - |\Lambda|$$

since D is SNC, this count $\# \Delta_g \cap \Delta_h$

$$e_c(D_g \setminus \Lambda) = e(D_g) - \sum_{\substack{\Delta_g \neq \Delta_h \\ \langle g \rangle \cap \langle h \rangle \neq 1}} \Delta_g \cdot \Delta_h$$



so

$$\sum_{g\in G}\left(1-\frac{1}{|g|}\right)e_c(D_g\setminus\Lambda) = \sum_{g\in G}\left(1-\frac{1}{|g|}\right)e(D_g) - \frac{1}{2}\sum'_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1}}\left(\left(1-\frac{1}{|g|}\right)+\left(1-\frac{1}{|h|}\right)\right)\Delta_g\cdot\Delta_h$$

Finally, we observe $D_g\setminus\Lambda\cdot D_h\setminus\Lambda = \begin{cases} D_g\cdot D_h & \text{if } \langle g\rangle\cap\langle h\rangle = 1_G \\ 0 & \text{if } \langle g\rangle\cap\langle h\rangle\neq 1_G\end{cases}$

by construction of Λ. Thus

$$\frac{1}{2}\sum'_{g\neq h}\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)D_g\setminus\Lambda\cdot D_h\setminus\Lambda = \frac{1}{2}\sum'_{\substack{\Delta_g\le D_g,\,\Delta_h\le D_h\\ \Delta_g\neq\Delta_h}}\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)\Delta_g\cdot\Delta_h - \frac{1}{2}\sum'_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1_G}}\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)\Delta_g\cdot\Delta_h$$

Putting everything together we obtain: $\langle g\rangle\cap\langle h\rangle\neq 1_G$

$$e_c(X^\circ) = |G|\left[e(Y) - \sum_{g\in G}\left(1-\frac{1}{|g|}\right)e(D_g) + \frac{1}{2}\sum'_{\Delta_g\neq\Delta_h}\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)\Delta_g\cdot\Delta_h\right]$$

$$-|\Delta|\cdot|G| + \frac{1}{2}\left[\sum'_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1_G}}\left(\underbrace{\overset{x}{\left(1-\frac{1}{|g|}\right)}+\overset{y}{\left(1-\frac{1}{|g|}\right)}-\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)}_{x-xy+y}\right)\Delta_g\cdot\Delta_h\right]|G|$$

$$x - xy + y \;=\; 1-(1-x)(1-y) = 1-\frac{1}{|g||h|}$$

$$\frac{1}{2}\sum_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1_G}}\Delta_g\cdot\Delta_h - \frac{1}{2}\sum_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1_G}}\frac{1}{|g||h|}\Delta_g\cdot\Delta_h$$

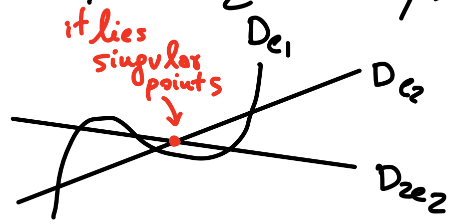
$$= |\Lambda| - \frac{1}{2}\sum_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1_G}}\frac{\Delta_g\cdot\Delta_h}{|g|\cdot|h|}$$

$$= |G|\left[e(Y) - \sum_{g\in G}\left(1-\frac{1}{|g|}\right)e(D_g) + \frac{1}{2}\sum'_{\Delta_g\neq\Delta_h}\left(1-\frac{1}{|g|}\right)\left(1-\frac{1}{|h|}\right)\Delta_g\cdot\Delta_h\right]$$

$$-\frac{1}{2}\sum'_{\substack{\Delta_g\neq\Delta_h\\ \langle g\rangle\cap\langle h\rangle\neq 1_G}}\frac{|G|}{|g|\cdot|h|}\Delta_g\cdot\Delta_h \qquad \blacksquare$$

Example We consider $\pi: X \longrightarrow \mathbb{P}^2$ with group $G = \mathbb{Z}_3^2 = \langle e_1, e_2 \rangle$

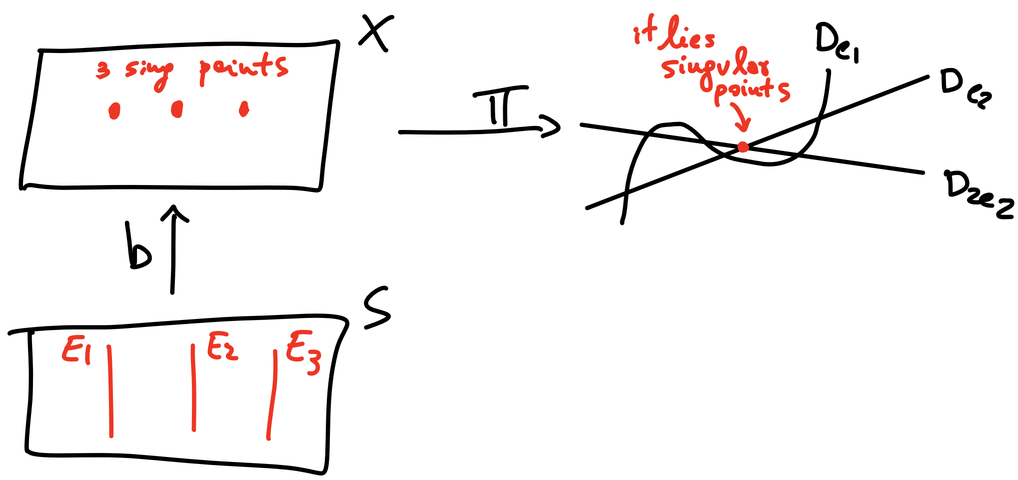
D_{e_1} = cubic, D_{e_2} = line, D_{2e_2} = line



The points over $D_{e_2} \cap D_{2e_2}$ are $\frac{|G|}{|\langle e_2, 2e_2 \rangle|} = \frac{3^2}{3} = 3$, and their type of singularities is:

$2e_2 = 2 \cdot e_2$ so $a = 2$ and then we have $\frac{1}{3}\left(1, 3 - \frac{3 \cdot 2}{3}\right) = \frac{1}{3}(1,1)$

The continued fraction of $\frac{3}{1}$ is $[3]$, so the HJ string of each of these 3-points consists of only an smooth rational curve with self-intersection -3



Let us determine the invariants:

$$q(S) = \sum_{\chi \in G^*} h^1(\mathbb{P}^2, -\mathcal{L}_\chi) = 0,$$

$$3K_S \equiv \tilde{\pi}^*\left((-3 \cdot 3 + 2 \cdot 5)H\right) - \left[3\left(1-\tfrac{2}{3}\right) \cdot E_1 + 3\left(1-\tfrac{2}{3}\right)E_2 + 3\left(1-\tfrac{2}{3}\right)E_3\right]$$

$$= \tilde{\pi}^* H - E_1 - E_2 - E_3 \Rightarrow 3K_S \equiv \tilde{\pi}^* H - E_1 - E_2 - E_3$$

so $9K_S^2 = (\underbrace{\tilde{\pi}^* H}_{9})^2 + 3\cdot(-3) = 0 \Rightarrow K_S^2 = 0.$

$$e(S) = 9\left[3 - \left(1-\tfrac{1}{3}\right)(0+2+2) + \left(1-\tfrac{1}{3}\right)^2 \cdot 7\right] + \left(\frac{9}{3} - \frac{9}{3\cdot 3}\right)\cdot 1$$
$$+ 3\cdot 1$$
$$= 9\left(3 - \frac{8}{3} + \frac{4}{9}\cdot 7\right) + 5 = 9\left(3 + \frac{4}{9}\right) = 27 + 9$$
$$= 36$$

Furthermore, by Noether's Formula

$$12\chi = K^2 + e = 36 \Rightarrow \chi = 3$$

so $p_g(S) = \chi + q - 1 = 2 \Rightarrow p_g(S) = 2.$

It is not so difficult to prove that S is a minimal properly elliptic surface (with Kodaira dimension 1) and the canonical map

$$\varphi_{|K_S|} : S \longrightarrow \mathbb{P}^1$$

is a morphism with a generic fibre that is a smooth elliptic curve.

Remark 1) One can also choice to compute p_g and χ directly using the above formulae but this can be done only once we have first computed all degrees l_χ, $\chi \in G^*$.

2) If one applies the above inequality for the Kodaira dimension of S, then

$$\kappa(S) \leq \kappa(\mathbb{P}^2, (-27+6\cdot 5)H)$$
$$= \kappa(\mathbb{P}^2, 3H)$$
$$= 2$$

so we would not get useful information. One can use other methods to prove $\kappa(S)=1$, so S is properly elliptic.

3) We remind that for $Y = \mathbb{P}^2$, then

$$\kappa(\mathbb{P}^2, dH) \in \{-\infty, 0, 2\}$$

can not be equal to 1.

Thus, smooth ab. coverings $\pi: X \to \mathbb{P}^2$ can not have Kodaira dim. 1 because

$$\kappa(X) = \kappa\Big(\mathbb{P}^2, |G|\Big(-3 + \sum_{g\in G} \frac{|g|-1}{|g|} d_g\Big)H\Big)$$

However, when $\pi: X \to \mathbb{P}^2$ is singular, such as the above example, then we may reach $\kappa(X)=1$.

The End